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ASYMPTOTIC SOLUTION OF THE PROBLEM OF THE PRESSURE OF A RIGID BODY ON A MEMBRANE†

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The problem of the pressure on a membrane of a punch in the form of an elliptic paraboloid is investigated assuming that the contact area is small. By means of the method of matched asymptotic expansions, the problem of unilateral contact is developed for an inner asymptotic expansion, the solution of which is based on results for the case of an infinite membrane [1]. The influence of the boundary of the membrane is modelled in formulating the asymptotic conditions at infinity. The relation between the force impressing the punch and its motion is determined. The sensitivity of the parameters of the elliptic contact area to the dimensions of the membrane and the position of the centre of the punch is investigated. A refined asymptotic model of contact interaction is proposed. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM AND ITS DISCUSSION

Suppose a membrane Ω, fastened along its contour $\partial\Omega$ and under uniform tension T, is pressed by a punch in the form of an elliptic paraboloid

$$\Phi(\mathbf{x}^0;\mathbf{x}) = (2R_1)^{-1}(x_1 - x_1^0)^2 + (2R_2)^{-1}(x_2 - x_2^0)^2 \tag{1.1}$$

where R_1 and R_2 are the radii of curvature of the main normal sections of the surface of the punch at its apex $\mathbf{x}^0 \in \Omega$. The translational displacement of the punch will be denoted by δ_0.

The lackling function of the membrane satisfies the problem (see, for example, [2,3])

$$-T\Delta_x u(\mathbf{x}) \geq 0, \quad u(\mathbf{x}) \geq \delta_0 - \Phi(\mathbf{x}^0;\mathbf{x}) \tag{1.2}$$

$$\Delta_x u(\mathbf{x})[u(\mathbf{x}) - \delta_0 + \Phi(\mathbf{x}^0;\mathbf{x})] = 0, \quad \mathbf{x} = (x_1, x_2) \in \Omega$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega \tag{1.3}$$

We will investigate problem (1.2), (1.3) on the assumption that the contact area Σ is small. We will designate ε as a small positive parameter and assume that

$$R_1 = \varepsilon R_1^*, \quad R_2 = \varepsilon R_2^*; \quad \delta_0 = \varepsilon\delta_0^* \tag{1.4}$$

where δ_0^* and R_1^* and R_2^* are comparable with the distance d_0 from the point $\mathbf{x}^0$ to the boundary $\partial\Omega$.

The contact area Σ [where the equality sign holds in the second inequality of (1.2)] is not known in advance. With certain constraints (see [2, Chapter 5, Sections 3 and 6]) it is possible *a priori* to assert that Σ is a simply connected region with a smooth boundary $\partial\Sigma$. In accordance with the adopted shape of the punch (1.1), the pressure $p(x_1, x_2) = T\Delta_x\Phi(\mathbf{x}^0; \mathbf{x})$ transferred by the punch to the membrane is uniform:

$$p = T(R_1 + R_2)(R_1R_2)^{-1} \tag{1.5}$$

Remark 1. By virtue of the maximum principle (see [4, Section 2.2]), the relation $u(\mathbf{x}) \geq 0$ results from the first inequality of (1.2), taking condition (1.3) into account. This inequality enables is to obtain an estimate for the contact area (the coincidence factor [2]). Thus, taking into account the second inequality of (1.2), we find that the region Σ is undoubtedly encompassed by an ellipse with centre at the point $\mathbf{x}^0$ and with semiaxes $\sqrt{(2\delta_0R_1)}$ and $\sqrt{(2\delta_0R_2)}$. Consequently, under condition (1.4), the size of the contact area will be of the order of εd_0.

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Earlier [1], an accurate solution of the contact problem for an infinite membrane was obtained. The semiaxes $c_0(1 + m)$ and $c_0(1 - m)$ of the elliptic contact area are expressed in terms of the force P acting on the punch

$$c_0^2 = \frac{P(R_1 + R_2)}{4\pi T}, \quad m = \frac{R_1 - R_2}{R_1 + R_2}; \quad P = \iint\limits_{\Sigma} p(y_1, y_2) dy_1 dy_2 \tag{1.6}$$

In an attempt to use formulae (1.6) in the case of a finite membrane, the question arises of how the force P depends on the displacement δ_0. Note also that, generally speaking, the centre of the contact area is displaced in relation to the apex of the punch and, consequently, to ensure a vertical position of the punch, it is necessary to apply to it the moments

$$M_1 = \iint\limits_{\Sigma} (y_2 - x_2^0) p(y_1, y_2) dy_1 dy_2, \quad M_2 = -\iint\limits_{\Sigma} (y_1 - x_1^0) p(\mathbf{y}) d\mathbf{y} \tag{1.7}$$

Substituting expression (1.5) into the final equality of (1.6) and relations (1.7), we find

$$P = p|\Sigma|; \quad M_1 = p|\Sigma|(x_2^c - x_2^0), \quad M_2 = -p|\Sigma|(x_1^c - x_1^0) \tag{1.8}$$

where $|\Sigma|$ is the area of the required contact region, and x_1^c and x_2^c are the coordinates of its centre of gravity. It is clear that the quantities mentioned should depend on the shape and dimensions of the membrane and the position of the punch.

Problem (1.2), (1.3) was studied within the framework of the theory of variational inequalities [2, 5, 6]. Algorithms for its numerical solution have been proposed ([7, 8], etc.). High-quality methods have been developed ([9], etc.). An asymptotic solution of the problem of the equilibrium of a loaded membrane supported by several small, plane, unilateral supports was obtained in [1]. Below, we use the method of matched asymptotic expansions (see [11–13], etc.), confirmed in [14] for the unilateral contact problem for a three-dimensional elastic body. The solution of the problem for a boundary layer is constructed using results obtained earlier [1].

2. EXTERNAL AND INTERNAL ASYMPTOTIC REPRESENTATIONS

We will denote Green's function with a pole at the point $\mathbf{x}^0$ by $G(\mathbf{x}^0; \mathbf{x})$. When $\mathbf{x} \to x^0$, the following asymptotic formula holds

$$G(\mathbf{x}^0; \mathbf{x}) = \frac{1}{2\pi} \ln \frac{r_0}{|\mathbf{x} - \mathbf{x}^0|} + o(1) \tag{2.1}$$

where r_0 is a constant having the dimension of length.

Remark 2. In the case of a simply-connected region Ω, the representation $G(\mathbf{x}^0; \mathbf{x}) = -(2\pi)^{-1} \ln |f(z^0; z)|$ is known (see, for example [15, Section 43]). Here, $z = x_1 + ix_2$ is a complex variable, and $\zeta = f(z^0; z)$ is the conformal mapping of region Ω onto the interior of the unit circle $|\zeta| < 1$, with $f(z^0; z^0) = 0$. Since the function $(z - z^0)^{-1} f(z^0; z)$ has, at the point z^0, a removable singularity and

$$\lim_{z \to z^0} \frac{f(z^0; z)}{z - z^0} = \left. \frac{df(z^0; z)}{dz} \right|_{z = z^0} \equiv f'(z^0; z^0) \neq 0$$

formula (2.1) acquires the form

$$G(\mathbf{x}^0; \mathbf{x}) = -(2\pi)^{-1} \ln(|z - z^0| |f'(z^0; z^0)|) + o(1)$$

The quantity $r_0 = |f'(z^0; z^0)|^{-1}$ is identical with the (internal) conformal radius of the region Ω with respect to the point $z^0 = x_1^0 + ix_2^0$ (see [16, Section 1.3] and [17, Section IV]).

Far from the contact area, we will unite the asymptotic representation of the solution of problem (1.2), (1.3) in the form

$$v(\mathbf{x}) = \frac{P}{T} G(\mathbf{x}^0; \mathbf{x}) \tag{2.2}$$

On the basis of an analysis of the axisymmetrical problem, in view of (1.4) we can write

$$P = \varepsilon P^* \tag{2.3}$$

In the neighbourhood of the contact area $\Sigma(\varepsilon)$ we will introduce the "extended" coordinates

$$\boldsymbol{\xi} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}^0) \tag{2.4}$$

Then, if we take into account the new scale, the distance from the apex of the punch to the edge of the membrane becomes equal to $\varepsilon^{-1}d_0$. Hence, the problem for the inner asymptotic expansion is formulated over the entire plane.

Relations (1.2) give

$$\begin{gathered}
-\Delta_\xi w(\varepsilon;\boldsymbol{\xi}) \geqslant 0, \quad w(\varepsilon;\boldsymbol{\xi}) \geqslant \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi})) \\
\Delta_\xi w(\varepsilon;\boldsymbol{\xi})[w(\varepsilon;\boldsymbol{\xi}) - \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi}))] = 0, \quad \boldsymbol{\xi} \in \mathbf{R}^2 \\
\Phi^*(\boldsymbol{\xi}) = (2R_1^*)^{-1}\xi_1^2 + (2R_2^*)^{-1}\xi_2^2
\end{gathered} \tag{2.5}$$

Boundary condition (1.2) is replaced by the asymptotic condition of the behaviour of $w(\varepsilon;\,\boldsymbol{\xi})$ as $|\boldsymbol{\xi}| \to \infty$, which we obtain as a result of matching with (2.2).

In the matching zone, where $\sqrt{(\varepsilon)}d_0/2 \leqslant |\mathbf{x} - \mathbf{x}^0| \leqslant \sqrt{(\varepsilon)}d_0$ or, which is the same, $d_0/(2\sqrt{\varepsilon}) \leqslant |\boldsymbol{\xi}| \leqslant d_0/\sqrt{\varepsilon}$, after substituting of (2.1) and (2.3) into (2.2) and replacement of the coordinates inverse to (2.4), we have

$$v(\varepsilon;\mathbf{x}^0 + \varepsilon\boldsymbol{\xi}) = \frac{\varepsilon P^*}{T}\left(\frac{1}{2\pi}\ln\frac{r_0}{\varepsilon|\boldsymbol{\xi}|} + O(\sqrt{\varepsilon})\right) \tag{2.6}$$

Therefore, formula (2.5) is completed by the following

$$w(\varepsilon;\boldsymbol{\xi}) = \frac{\varepsilon P^*}{2\pi T}\ln\frac{r_0}{\varepsilon|\boldsymbol{\xi}|} + O(|\boldsymbol{\xi}|^{-1}), \quad |\boldsymbol{\xi}| \to \infty \tag{2.7}$$

Thus, the construction of the asymptotic form of the initial problem to a first approximation has been reduced to solving the problem of unilateral contact for the boundary layer (2.5), (2.7) and, in addition, to expressing the quantity P^* in terms of δ_0^*.

3. THE EQUATION RELATING THE MOTION OF THE PUNCH WITH THE FORCE ACTING ON IT

The solution of contact problem (2.5), (2.7) for an infinite membrane will be sought in the form

$$w(\varepsilon;\,\boldsymbol{\xi}) = \varepsilon W(\boldsymbol{\xi}) \tag{3.1}$$

Denoting by Σ^* the corresponding contact area with the boundary $\partial\Sigma^*$, we replace the unilateral contact condition (2.5) (see [2]) with the relations

$$\Delta_\xi W(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbf{R}^2 / \overline{\Sigma}^* \tag{3.2}$$

$$W(\boldsymbol{\xi}) = \delta_0^* - \Phi^*(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \partial\Sigma^* \tag{3.3}$$

$$\partial_i W(\boldsymbol{\xi}) = -(R_i^*)^{-1}\xi_i \quad (i = 1,2), \quad \boldsymbol{\xi} \in \partial\Sigma^* \tag{3.4}$$

The condition at infinity is rewritten as

$$W(\boldsymbol{\xi}) = \frac{P^*}{2\pi T}\ln\frac{r_0}{\varepsilon|\boldsymbol{\xi}|} + O(|\boldsymbol{\xi}|^{-1}), \quad |\boldsymbol{\xi}| \to \infty \tag{3.5}$$

The solution of problem (3.2)–(3.5) was expressed [1] in terms of the complex potential

$$W(\xi) = \operatorname{Re}\varphi(\xi_1 + i\xi_2) \tag{3.6}$$

It has been established that the region Σ^* is bounded by an ellipse, where its enlargement to the expanded complex plane represents the form of the exterior of the unit circle in conformal mapping:

$$\xi_1 + i\xi_2 = \omega(\zeta), \quad \omega(\zeta) = c_0^*\zeta + c_1^*\zeta^{-1} \tag{3.7}$$

$$(c_0^*)^2 = (4\pi T)^{-1} P^*(R_1^* + R_2^*), \quad c_1^* = c_0^*(R_1^* - R_2^*)(R_1^* + R_2^*)^{-1} \tag{3.8}$$

The following expression is obtained for the derivative of the complex potential.

$$\varphi'[\omega(\zeta)] = c_2\zeta^{-1}; \quad c_2 = -2c_0^*(R_1^* + R_2^*)^{-1} \tag{3.9}$$

By means of the formula

$$\varphi[\omega(\zeta)] = \int \varphi'[\omega(\zeta)]\frac{d\omega(\zeta)}{d\zeta}d\zeta$$

using relations (3.7) and (3.9), we obtain

$$\varphi[\omega(\zeta)] = c_2\left(c_0^* \ln\zeta + \frac{c_1^*}{2}\frac{1}{\zeta^2} + c_3^*\right) \tag{3.10}$$

We determine the integration constant c_3^*, satisfying boundary condition (3.3) with $|\zeta| = 1$. As a result, we obtain

$$c_3^* = \delta_0^* - (c_0^*)^2(R_1^* + R_2^*)^{-1} \tag{3.11}$$

Thus, according to (3.10) and (3.7), the function (3.8) has the following behaviour at infinity

$$W(\xi) = -c_2c_0^* \ln(c_0^* |\xi^{-1}|) + c_3^* + O(|\xi|^{-2}), \quad |\xi| \to \infty \tag{3.12}$$

Comparing (3.12) and (2.7), taking (3.8), (3.11) and (3.1) into account, we derive

$$\delta_0^* - \frac{(c_0^*)^2}{R_1^* + R_2^*} = \frac{P^*}{2\pi T}\ln\frac{r_0}{\varepsilon c_0^*} \tag{3.13}$$

Recalling the assumptions (1.4) and (2.3), we combine (3.13) and (3.8) into the single equation

$$\frac{P}{4\pi T}\left(1 + \ln\frac{4\pi T r_0^2}{P(R_1 + R_2)}\right) = \delta_0 \tag{3.14}$$

Equation (3.14) is used to determine the force P in terms of the displacement δ_0.
We will introduce the dimensionless quantities

$$\tilde{P} = \frac{P}{4\pi T r_0}, \quad \tilde{\delta}_0 = \frac{\delta_0}{r_0}, \quad \varepsilon = \frac{R_1 + R_2}{r_0}$$

Then, Eq. (3.14) takes the form

$$\tilde{P} + \Lambda^{-1}\tilde{P}\ln\tilde{P} = \Lambda^{-1}\tilde{\delta}_0, \quad \Lambda = 1 + \ln(1/\varepsilon) \tag{3.15}$$

Equation (3.15) contains a large parameter Λ and in turn allows of an asymptotic solution (see [18, Chapter 1, Section 5]).

Finally, the contact area Σ is close to elliptic. Its semiaxes to a first approximation are calculated by means of formulae (1.6), where their arithmetic mean is $c_0 = \varepsilon c_0^*$.

It is noteworthy that, in the axisymmetric problem, formula (3.14), expressing the dependence of the motion of the punch on the force acting on it, agrees with the accurate formula.

4. MOMENTS OF THE SYSTEM OF LOADS THAT MAINTAINS THE PUNCH IN THE VERTICAL POSITION

In the problem for the boundary layer (2.5), (2.7), the influence of the shape and dimensions of the membrane is taken into account in the formulation of asymptotic condition (2.7). The latter was obtained from formula (2.1), which can easily be made more precise. We have

$$G(\mathbf{x}^0;\mathbf{x}) = \frac{1}{2\pi}\ln\frac{r_0}{|\mathbf{x}-\mathbf{x}^0|} + \sum_{i=1}^{2} B_i(x_i - x_i^0) + \sum_{i,j=1}^{2} C_{ij}(x_i - x_i^0)(x_j - x_j^0) + \tag{4.1}$$

$$+O(|\mathbf{x}-\mathbf{x}^0|^3), \quad \mathbf{x}\to\mathbf{x}^0$$

The quantities B_1, B_2 and $C_{21} = C_{12}$, $C_{22} = -C_{11}$, depending on the position of the point $\mathbf{x}^0$, have dimensions of L^{-1} and L^{-2}, respectively (L is the dimension of length).

Taking into account the first two terms of the sum (4.1), instead of (2.7) we can write

$$w(\varepsilon;\xi) = \varepsilon^2 V^*(\xi) + \frac{\varepsilon P^*}{2\pi T}\ln\frac{r_0}{\varepsilon|\xi|} + O(|\xi|^{-1}), \quad |\xi|\to\infty \tag{4.2}$$

$$V^*(\xi) = \frac{P^*}{T}\sum_{i=1}^{2} B_i\xi_i \tag{4.3}$$

The solution of problem (2.5), (4.2) will be presented in the form

$$w(\varepsilon;\xi) = \varepsilon^2 V^*(\xi) + \varepsilon W(\xi) \tag{4.4}$$

The function W satisfies relation (3.5) and the following relations

$$\begin{gathered} -\Delta_\xi W(\xi) \geq 0, \quad W(\xi) \geq \delta_0^* - \Phi^*(\xi) - \varepsilon V^*(\xi) \\ \Delta_\xi W(\xi)[W(\xi) - \delta_0^* + \Phi^*(\xi) + \varepsilon V^*(\xi)] = 0, \quad \xi\in\mathbf{R}^2 \end{gathered} \tag{4.5}$$

Since the right-hand side of the second inequality of (4.5) is a second-degree polynomial, the solution of problem (4.5), (3.5) is constructed using well-known results [1]. Isolating complete squares, we obtain

$$\Phi^*(\xi) + \varepsilon V^*(\xi) = \sum_{i=1}^{2} (2R_i^*)^{-1}(\xi_i - \xi_i^c)^2 + O(\varepsilon^2) \tag{4.6}$$

$$\xi_i^c = -\varepsilon P^* T^{-1} R_i^* B_i \quad (i=1,2) \tag{4.7}$$

Since, when formulating (4.2), after the replacement of (2.4), only terms $O(\varepsilon|\xi|)$ are retained in expansion (4.1) [see (4.3)], in formula (4.6) the constant of a higher order of smallness is not taken into account.

Thus, within the accuracy adopted, the centre of the contact area is displaced to a point with coordinates (4.7) or [we return to the real scale, recalling formulae (1.5), (2.3) and (2.4)]

$$x_i^c = x_i^0 - PT^{-1}R_iB_i \quad (i=1,2) \tag{4.8}$$

The dimensions of the contact area Σ are defined as before by formulae (1.6), where, in the main, $|\Sigma| = \pi c_0^2(1-m^2)$. Dependence (3.14) of the motion on the force does not change. Moments (1.7) are found by means of formulae (1.8) and (4.8) in the form

$$M_1^* = P^*\xi_2^c, \quad M_2^* = -P^*\xi_1^c \tag{4.9}$$

$$M_i = \varepsilon^2 M_i^* \quad (i = 1,2) \tag{4.10}$$

Finally, instead of (3.12), we obtain the following expansion

$$W(\xi) = \frac{P^*}{2\pi T} \ln \frac{c_0^*}{|\xi|} + \frac{P^*}{2\pi T} \frac{\xi_1^c \xi_1 + \xi_2^c \xi_2}{\xi_1^2 + \xi_2^2} + O(|\xi|^{-2}), \quad |\xi| \to \infty \tag{4.11}$$

5. REFINEMENT OF THE CONSTRUCTION OF THE ASYMPTOTIC FORM

Formula (4.11) dictates the renewal of external asymptotic representation (2.2). We will assume that

$$v(\mathbf{x}) = \frac{P}{T} G(\mathbf{x}^0; \mathbf{x}) + \sum_{i=1}^{2} \frac{M_i}{T} G^{(i)}(\mathbf{x}^0; \mathbf{x}). \tag{5.1}$$

Here, solutions are introduced for problems of the action on the membrane of concentrated moments

$$G^{(1)}(\mathbf{x}^0; \mathbf{x}) = \frac{1}{2\pi} \frac{x_2 - x_2^0}{|\mathbf{x} - \mathbf{x}^0|^2} + g^{(1)}(\mathbf{x}^0; \mathbf{x}), \tag{5.2}$$

$$G^{(2)}(\mathbf{x}^0; \mathbf{x}) = -\frac{1}{2\pi} \frac{x_1 - x_1^0}{|\mathbf{x} - \mathbf{x}^0|^2} + g^{(2)}(\mathbf{x}^0; \mathbf{x}),$$

$$g^{(i)}(\mathbf{x}^0; \mathbf{x}) = A_0^{(i)} + O(|\mathbf{x} - \mathbf{x}^0|), \quad \mathbf{x} \to \mathbf{x}^0 \quad (i = 1,2) \tag{5.3}$$

Using (4.1) and (5.3) and taking (2.3) and (4.10) into account, we formulate the following asymptotic condition for the boundary layer [cf. (4.2) and (4.3)]

$$w(\varepsilon; \xi) = \varepsilon^2 V^*(\varepsilon; \xi) + \frac{\varepsilon P^*}{2\pi T} \ln \frac{c_0^*}{|\xi|} + \frac{\varepsilon}{2\pi T} \frac{-M_2^* \xi_1 + M_1^* \xi_2}{\xi_1^2 + \xi_2^2} + O(|\xi|^{-2}), \quad |\xi| \to \infty \tag{5.4}$$

$$V^*(\varepsilon; \xi) = \frac{P^*}{T} \left(\sum_{i=1}^{2} B_i \xi_i + \varepsilon \sum_{i,j=1}^{2} C_{ij} \xi_i \xi_j \right) + \sum_{i=1}^{2} \frac{M_i^*}{T} A_0^{(i)} \tag{5.5}$$

Note that formulae (5.4) and (5.5) are derived by retaining previously unused terms (of higher order of smallness) in expansions (4.1) and (5.3). Here, use is made of information that the values of M_i^*, according to relations (4.9) and (4.7), are of the order of ε.

Thus, for inner asymptotic expansion (4.4), problem (2.5), (5.4) is obtained. We will write its asymptotic solution. Since, generally speaking, the coefficient C_{12} is non-zero, the elliptic contact area is turned by a certain angle φ with respect to the coordinate axes. If $R_1^* = R_2^*$, then φ is determined by the quadratic form

$$\sum_{i,j=1}^{2} C_{ij} \xi_i \xi_j$$

If, for example, $R_1^* > R_2^*$, then, apart from terms of the order of ε^3 [formulae (5.4) and (5.5) are of this accuracy]

$$\varphi = -\varepsilon^2 \frac{2R_1^* R_2^*}{R_1^* - R_2^*} \frac{P^*}{T} C_{12} \tag{5.6}$$

The coordinates (4.8) of the centre of the contact area at a given stage are not refined. Its semiaxes $\varepsilon c_0^*(1 + m)$ and $\varepsilon c_0^*(1 - m)$ are calculated by means of the formulae [refining (1.6)]

$$(c_0^*)^2 = \frac{P^*}{4\pi T}(R_1^* + R_2^*) - \varepsilon^2 \frac{(P^*)^2}{2\pi T}[(R_1^*)^2 - (R_2^*)^2]C_{11} \tag{5.7}$$

$$m = \frac{R_1^* - R_2^*}{R_1^* + R_2^*} - \varepsilon^2 \frac{4R_1^* R_2^*}{R_1^* - R_2^*}\frac{P^*}{T}C_{11} \tag{5.8}$$

Within the accuracy adopted, the values of M_i^* are identical to (4.9). To determine P^*, neglecting unimportant terms, we derive the equation

$$\frac{P^*}{4\pi T}\left(1 + \ln\frac{4\pi T r_0^2}{\varepsilon^2 P^*(R_1^* + R_2^*)}\right) + \tag{5.9}$$
$$+\varepsilon^2 (P^*) T^{-2}[R_1^* B_1 A_0^{(2)} - R_2^* B_2 A_0^{(1)} - (2\pi)^{-1}(R_1^* - R_2^*)C_{11}] = \delta_0^*$$

Thus, the refined relation between the force acting on the punch and its motion, according to (5.9), (1.4) and (2.3), is as follows:

$$\frac{P}{4\pi T}\left(1 + \ln\frac{4\pi T r_0^2}{P(R_1 + R_2)}\right) + \frac{P^2}{T^2}\left[R_1 B_1 A_0^{(2)} - R_2 B_2 A_0^{(1)} - \frac{1}{2\pi}(R_1 - R_2)C_{11}\right] = \delta_0 \tag{5.10}$$

Example. We will calculate the coefficients in relations (5.6)–(5.10) for a circular membrane of radius a. Using the conformal transformation of the circle to the unit circle, transferring the point $\mathbf{x}^0$ to the centre of the latter (see, for example [15, Section 32]), we find (for the notation, see Remark 2)

$$G(\mathbf{x}^0;\mathbf{x}) = \frac{1}{2\pi}\ln\frac{r_0}{|z - z^0|} + \frac{1}{2\pi}\ln\frac{|a^2 - \bar{z}^0 z|}{a r_0}; \quad r_0 = a\left(1 - \frac{|z^0|^2}{a^2}\right)$$

Hence, the coefficients in (4.1) are determined

$$B_i = -\frac{1}{2\pi}\frac{x_i^0}{a r_0}; \quad C_{11} = -C_{22} = -\frac{1}{2\pi}\frac{(x_1^0)^2 - (x_2^0)^2}{2a^2 r_0^2}, \quad C_{12} = -\frac{1}{2\pi}\frac{x_1^0 x_2^0}{a^2 r_0^2}$$

The constant $A_0^{(i)} = g^{(i)}(\mathbf{x}^0; \mathbf{x}^0)$ is sought as the arithmetic mean of the boundary values of the function $g^{(i)}(\mathbf{x}^0; \mathbf{x})$ after they have been conformally transformed to the unit circle [see (5.2)]. As a result of simple calculations, we have

$$A_0^{(1)} = -\frac{1}{2\pi}\frac{x_2^0}{a r_0}, \quad A_0^{(2)} = \frac{1}{2\pi}\frac{x_1^0}{a r_0}$$

6. REMARKS

An attempt at subsequent complication of the construction of the asymptotic form leads to the appearance in the solution of the problem for the boundary layer of a deviation of the shape of the contact area from elliptic. A study of the variation of the contact area was undertaken [19, 20].

Formulae (5.6)–(5.8) indicate the sensitivity of the parameters of the contact area to the dimensions of the membrane and the position of the centre of the punch. Remember that the radius r_0 and the quantities B_i, $A_0^{(i)}$ and C_{ij} have dimensions of L, L^{-1} and L^{-2}, respectively.

Note that, within the framework of the asymptotic model constructed [see, in particular, formulae (4.1), (4.7), (4.9) and (3.14), (5.10)], the equilibrium position of a small sphere on a horizontal weightless membrane is the point of the local maximum of the internal conformal radius (see also [16, Section 1.3]).

If the region Ω is not simply connected, then for Green's function there is no simple relation with the conformal mapping (see [21, Section 223] and [22, Section 3, Chapter VI]). Nonetheless, the value of r_0 (the harmonic radius of the region Ω will respect to the point $\mathbf{x}^0$) inherits [23] the main properties of the conformal radius (see [17, Section IV]).

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